

ABSORPTION TIME AND TREE LENGTH OF THE KINGMAN COALESCENT AND THE GUMBEL DISTRIBUTION

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Abstract

Formulas are provided for the cumulants and the moments of the time T back to the most recent common ancestor of the Kingman coalescent. It is shown that both the j th cumulant and the j th moment of T are linear combinations of the values $\zeta(2m)$, $m \in \{0, \dots, \lfloor j/2 \rfloor\}$, of the Riemann zeta function ζ with integer coefficients. The proof is based on a solution of a two-dimensional recursion with countably many initial values. A closely related strong convergence result for the tree length L_n of the Kingman coalescent restricted to a sample of size n is derived. The results give reason to revisit the moments and central moments of the classical Gumbel distribution.

Keywords: absorption time; cumulants; Euler–Mascheroni integrals; Gumbel distribution; infinite convolution; Kingman coalescent; moments; most recent common ancestor; tree length; zeta function

Running head: Absorption time and tree length of the Kingman coalescent

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1 Introduction

Kingman’s coalescent [8, 9], the most important coalescent among the class of all exchangeable coalescents, is a continuous time Markov process $\Pi = (\Pi_t)_{t \geq 0}$ with state space \mathcal{P} , the set of partitions of $\mathbb{N} := \{1, 2, \dots\}$. If this process is in a state with b blocks, then by definition during each transition any two blocks merge together at rate 1. This process starts in the partition of \mathbb{N} into singletons and reaches its absorbing state, the partition consisting of the single block \mathbb{N} , in finite time almost surely. Recently there is much interest in certain functionals of coalescent processes (restricted to a sample of size $n \in \mathbb{N}$) such as the number of jumps, the absorption time, or the total tree length to mention a few of them. In this manuscript we provide some new results on the absorption time and the total tree length of the Kingman coalescent. These results give reason to revisit the classical Gumbel distribution.

The article is organized as follows. In the following Section 2 results on functionals of the Kingman coalescent, such as the absorption time and the tree length, are provided. Section 3 is devoted to the classical Gumbel distribution. We recall well known results of the Gumbel distribution but also shed some new light in particular on the central moments of this distribution. Proofs are provided in Section 4. The article finishes with an appendix where the first central moments of the Gumbel distribution are given explicitly. The appendix furthermore provides the spectral decomposition of the transition matrix of a pure death process having distinct death rates.

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2 Absorption time and tree length

Kingman [8] studied the absorption time T of Π . In the biological context T is called the time back to the most recent common ancestor or the age of the most recent common ancestor. It is well known that $T = \sum_{k=2}^{\infty} \tau_k$ is an infinite convolution of independent exponentially distributed random variables τ_k with parameter $\lambda_k := k(k-1)/2$. Using the inversion formula for Fourier transforms it is readily checked that T has a bounded and infinitely often differentiable density $g : \mathbb{R} \rightarrow [0, \infty)$ with respect to Lebesgue measure λ on \mathbb{R} . Kingman [8, p. 37, Eq. (5.9)] showed that $g(t) = \sum_{k=2}^{\infty} (-1)^k (2k-1) \lambda_k e^{-\lambda_k t}$, $t \in (0, \infty)$. It is furthermore known (Watterson [15, p. 213], Tavaré [13, p. 132] that T has mean $\mathbb{E}(T) = 2$ and variance $\text{Var}(T) = 4\pi^2/3 - 12 \approx 1.15947$ and, hence, second moment $\mathbb{E}(T^2) = 4\pi^2/3 - 8 \approx 5.15947$. To the best of the authors knowledge higher moments and cumulants of T have not been derived so far. Proposition 2.1 and Theorem 2.2 below provide full information on the cumulants and moments of T .

Proposition 2.1 *For the Kingman coalescent the absorption time T has cumulants*

$$\kappa_j(T) = (j-1)! 2^j \sum_{k=2}^{\infty} \frac{1}{k^j (k-1)^j}, \quad j \in \mathbb{N}, \quad (1)$$

and moments

$$\mathbb{E}(T^j) = j! 2^j \sum_{k=2}^{\infty} \frac{(-1)^k (2k-1)}{k^j (k-1)^j} = j! \sum_{m=1}^{\infty} \frac{1}{m^j} \left(\frac{4m-1}{(2m-1)^j} - \frac{4m+1}{(2m+1)^j} \right), \quad j \in \mathbb{N}. \quad (2)$$

In particular, $\kappa_j(T) \sim (j-1)!$ and $\mathbb{E}(T^j) \sim 3j!$ as $j \rightarrow \infty$. Alternatively,

$$\mathbb{E}(T^j) = j! \sum_{2 \leq k_1 \leq \dots \leq k_j} \frac{1}{\lambda_{k_1} \cdots \lambda_{k_j}} = j! 2^j \sum_{2 \leq k_1 \leq \dots \leq k_j} \prod_{i=1}^j \frac{1}{k_i (k_i - 1)}, \quad j \in \mathbb{N}. \quad (3)$$

Remarks. 1. For $n \in \mathbb{N}$ let T_n denote the absorption time of the Kingman coalescent restricted to a sample of size n . The proof of Proposition 2.1 provided in Section 4 shows that $T_n \rightarrow T$ almost surely as $n \rightarrow \infty$ with convergence of all moments, which implies (see, for example, [6, Proposition 3.12]) the convergence $T_n \rightarrow T$ in L^p for any $p \in (0, \infty)$. Moreover, the sequence $(T_n^p)_{n \in \mathbb{N}}$ is uniformly integrable for any $p \in (0, \infty)$.

2. Let $g_n : \mathbb{R} \rightarrow [0, \infty)$ denote the density of T_n . Using the inversion formula for Fourier transforms it is straightforward to establish the local convergence result $\sup_{t \in \mathbb{R}} |g_n(t) - g(t)| = O(1/n)$. Scheffé's theorem (see, for example, [1, Theorem 16.12]) implies that $g_n \rightarrow g$ in L^1 , and, therefore, $|\mathbb{P}(T_n \in B) - \mathbb{P}(T \in B)| = |\int_B g_n d\lambda - \int_B g d\lambda| \leq \int |g_n - g| d\lambda \rightarrow 0$ for all Borel sets $B \subseteq \mathbb{R}$.

3. With some more effort it can be even verified that $\sup_{t \in \mathbb{R}} |n(g_n(t) - g(t)) - 2g'(t)| = O(1/n)$. The proof of this result is again based on the inversion formula for Fourier transforms, however a bit technical and therefore omitted here. We will not use this advanced local convergence result in our further considerations.

The following result shows that the cumulants $\kappa_j(T)$ and the moments $\mathbb{E}(T^j)$ of T are related to the Riemann zeta function ζ . More precisely, $\kappa_j(T)$ and $\mathbb{E}(T^j)$ are both linear combinations of the zeta values $\zeta(2m)$, $m \in \{0, \dots, \lfloor j/2 \rfloor\}$, with integer coefficients.

Theorem 2.2 For all $j \in \mathbb{N}$,

$$\kappa_j(T) = (-1)^j 2^{j+1} \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{(2j-2m-1)!}{(j-2m)!} \zeta(2m) \quad (4)$$

and

$$\mathbb{E}(T^j) = (-1)^j j 2^{j+1} \sum_{m=0}^{\lfloor j/2 \rfloor} (2m-1) \left(1 - \frac{1}{2^{2m-1}}\right) \frac{(2j-2m-2)!}{(j-2m)!} \zeta(2m), \quad (5)$$

where ζ denotes the zeta function.

Remarks. 1. The coefficient in (5) in front of $\zeta(2m)$ is integer, since $2^{j+1}(1 - 1/2^{2m-1}) = 2^{j+1} - 2^{j-2m+2} \in \mathbb{Z}$ and $(2j-2m-2)!/(j-2m)! \in \mathbb{Z}$ for all $j \in \mathbb{N}$ and all $m \in \{0, \dots, \lfloor j/2 \rfloor\}$.
2. Let B_0, B_1, \dots denote the Bernoulli numbers defined recursively via $B_0 := 1$ and $B_n := -1/(n+1) \sum_{k=0}^{n-1} \binom{n+1}{k} B_k$ for $n \in \mathbb{N}$. For instance, $B_1 = -1/2$, and $B_2 = 1/6$. Since $\zeta(2m) = (-1)^{m-1} (2\pi)^{2m} / (2(2m)!) B_{2m}$ is a rational multiple of π^{2m} , Theorem 2.2 implies that $\kappa_j(T)$ and $\mathbb{E}(T^j)$ are both polynomials of degree $\lfloor j/2 \rfloor$ with rational coefficients evaluated at π^2 . The following tables are easily computed using (4) and (5).

j	$\kappa_j(T)$ in terms of $\zeta(\cdot)$	$\kappa_j(T)$ in terms of π	$\kappa_j(T)$ numerically
1	2	2	2.00000
2	$8\zeta(2) - 12$	$\frac{4}{3}\pi^2 - 12$	1.15947
3	$160 - 96\zeta(2)$	$160 - 16\pi^2$	2.08633
4	$192\zeta(4) + 1920\zeta(2) - 3360$	$\frac{32}{15}\pi^4 + 320\pi^2 - 3360$	6.07947
5	$-7680\zeta(4) - 53760\zeta(2) + 96768$	$-\frac{256}{3}\pi^4 - 8960\pi^2 + 96768$	24.10210

Table 1: The first five cumulants of T .

j	$\mathbb{E}(T^j)$ in terms of $\zeta(\cdot)$	$\mathbb{E}(T^j)$ in terms of π	$\mathbb{E}(T^j)$ numerically
1	2	2	2.00000
2	$8\zeta(2) - 8$	$\frac{4}{3}\pi^2 - 8$	5.15947
3	$96 - 48\zeta(2)$	$96 - 8\pi^2$	17.04317
4	$672\zeta(4) + 768\zeta(2) - 1920$	$\frac{112}{15}\pi^4 + 128\pi^2 - 1920$	70.63058
5	$-20160\zeta(4) - 19200\zeta(2) + 53760$	$-224\pi^4 - 3200\pi^2 + 53760$	357.62953

Table 2: The first five moments of T .

The proofs provided in Section 4 rely on the fact that the jump chain of the block counting process of the Kingman coalescent is deterministic, which implies that T is an infinite convolution of exponentially distributed random variables. The proof of Theorem 2.2 is based on a solution of a two-dimensional recursion with an infinite number of initial values (see Lemma 4.1).

Our methods do not seem to be directly applicable to (absorption times of) other exchangeable coalescent processes, since the jump chain of the block counting process of a coalescent with multiple collisions is in general not deterministic.

Our methods are partly useful to analyze further functionals of the Kingman n -coalescent (restricted to a sample of size $n \in \mathbb{N}$). As an example we provide detailed information on the tree

length L_n (the sum of the lengths of all branches of the n -coalescent tree) of the Kingman n -coalescent. Let G be a standard Gumbel distributed random variable with distribution function $x \mapsto \exp(-\exp(-x))$, $x \in \mathbb{R}$. It is well known (see, for example, [14, p. 22–23], [3, Lemma 7.1] or [4, Lemma 2.21]) that L_n has the same distribution as the maximum of $n - 1$ independent and exponentially distributed random variables with parameter $1/2$ and that $L_n/2 - \log n \rightarrow G$ in distribution as $n \rightarrow \infty$. We verify the following stronger convergence result.

Theorem 2.3 (Strong asymptotics of the tree length) *For the Kingman coalescent, as $n \rightarrow \infty$, $G_n := L_n/2 - \log n \rightarrow G$ almost surely and in L^p for any $p \in (0, \infty)$, where G is standard Gumbel distributed. Moreover, the sequence $(G_n^p)_{n \in \mathbb{N}}$ is uniformly integrable for any $p \in (0, \infty)$.*

Remarks. 1. Theorem 2.3 implies the convergence $\kappa_j(G_n) \rightarrow \kappa_j(G)$ of all cumulants and the convergence $\mathbb{E}(G_n^j) \rightarrow \mathbb{E}(G^j)$ of all moments, which is one of the starting points of the proof of Theorem 2.3. Formulas for the cumulants and the moments of L_n are provided in (25), (27) and (28).

2. For asymptotic results on the tree length L_n for Λ -coalescents with $\Lambda = \beta(a, b)$ being the beta distribution with parameters $a, b \in (0, \infty)$ we refer the reader to [7, Theorem 1] for $0 < a < 1$ and $b := 2 - a$ and to [3, Corollary 4.3 and Theorem 5.2] for $a = b = 1$ (Bolthausen–Sznitman coalescent). The asymptotics of L_n for Ξ -coalescents with dust is provided in [10, Theorem 3].

3 The Gumbel distribution revisited

Theorem 2.3 and its proof smooth the way to establish a result (Theorem 3.1) on the Gumbel distribution, also called the extreme value distribution of type 1. Recall that a standard Gumbel distributed random variable G has (see, for example, [5, p. 12, Eqs. (22.29) and (22.30)]) cumulants $\kappa_1 = \kappa_1(G) = \gamma$ (Euler’s constant) and $\kappa_j = \kappa_j(G) = (-1)^j \Psi^{(j-1)}(1) = (j-1)! \zeta(j)$, $j \geq 2$. Note that $\kappa_j \sim (j-1)!$ as $j \rightarrow \infty$. Moreover, G has moments

$$m_n := \mathbb{E}(G^n) = \int_0^\infty (-\log u)^n e^{-u} du = (-1)^n \Gamma^{(n)}(1), \quad n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}. \quad (6)$$

The integral in (6) is sometimes called the n th Euler–Mascheroni integral. In the analytic community its interpretation as the n th moment of the Gumbel distribution often remains unmentioned. It is well known that $m_n \sim n!$ as $n \rightarrow \infty$. The moments m_1, m_2, \dots can be recursively computed via the relation between cumulants and moments

$$m_n = \sum_{k=1}^n \binom{n-1}{k-1} \kappa_k m_{n-k} = \gamma m_{n-1} + (n-1)! \sum_{k=2}^n \frac{1}{(n-k)!} \zeta(k) m_{n-k}, \quad n \in \mathbb{N}, \quad (7)$$

in agreement with the recursion provided on top of p. 214 in the book of Boros and Moll [2]. A useful and well known formula for the moments is

$$m_n = \sum_{\pi \in \mathcal{P}_n} \prod_{B \in \pi} \kappa_{|B|}, \quad n \in \mathbb{N}, \quad (8)$$

where the sum extends over all partitions π of the set \mathcal{P}_n of all partitions of $\{1, \dots, n\}$, and the product has to be taken over all blocks B of π . Since there exist $n!/(a_1! \dots a_n! 1^{a_1} \dots n^{a_n})$

partitions $\pi \in \mathcal{P}_n$ having a_j blocks of size j , $1 \leq j \leq n$, the above formula can be also written as

$$m_n = n! \sum_{a_1, \dots, a_n} \prod_{i=1}^n \frac{1}{a_i!} \left(\frac{\kappa_i}{i!} \right)^{a_i} = n! \sum_{a_1, \dots, a_n} \frac{\gamma^{a_1}}{a_1!} \prod_{i=2}^n \frac{1}{a_i!} \left(\frac{\zeta(i)}{i} \right)^{a_i}, \quad n \in \mathbb{N}, \quad (9)$$

where the sum \sum_{a_1, \dots, a_n} extends over all $a_1, \dots, a_n \in \mathbb{N}_0$ satisfying $\sum_{i=1}^n i a_i = n$. For instance, $m_1 = \gamma$, $m_2 = \gamma^2 + \zeta(2)$, and $m_3 = \gamma^3 + 3\gamma\zeta(2) + 2\zeta(3)$. In particular, m_n is a polynomial of degree n in the variable $x := (x_1, \dots, x_n) := (\gamma, \zeta(2), \dots, \zeta(n))$ with nonnegative integer coefficients. In the following we focus on the central moments $m'_n := \mathbb{E}((G - \gamma)^n)$, $n \in \mathbb{N}_0$, of the Gumbel distribution. Clearly,

$$m_n = \mathbb{E}((G - \gamma + \gamma)^n) = \sum_{j=0}^n \binom{n}{j} \gamma^{n-j} m'_j, \quad n \in \mathbb{N}_0, \quad (10)$$

and

$$m'_n = \sum_{j=0}^n \binom{n}{j} (-\gamma)^j m_{n-j} = n! \sum_{j=0}^n \frac{(-\gamma)^j}{j!} \frac{m_{n-j}}{(n-j)!} \sim n! \sum_{j=0}^{\infty} \frac{(-\gamma)^j}{j!} = n! e^{-\gamma}$$

as $n \rightarrow \infty$. As for the moments we obtain for the central moments the recursion

$$m'_n = \sum_{k=1}^n \binom{n-1}{k-1} \kappa_k (G - \gamma) m'_{n-k} = (n-1)! \sum_{k=2}^n \frac{1}{(n-k)!} \zeta(k) m'_{n-k}, \quad n \in \mathbb{N},$$

with solution

$$m'_n = \sum_{\pi \in \mathcal{P}_n} \prod_{B \in \pi} \kappa_{|B|} (G - \gamma) = n! \sum_{a_2, \dots, a_n} \prod_{i=2}^n \frac{1}{a_i!} \left(\frac{\zeta(i)}{i} \right)^{a_i}, \quad n \in \mathbb{N}, \quad (11)$$

where the last sum \sum_{a_2, \dots, a_n} extends over all $a_2, \dots, a_n \in \mathbb{N}_0$ satisfying $\sum_{i=2}^n i a_i = n$. In particular, for all $n \geq 2$, m'_n is a polynomial of degree $\lfloor n/2 \rfloor$ in the variable $(\zeta(2), \dots, \zeta(n))$ with nonnegative integer coefficients. Theorem 3.1 below provides an alternative formula for m'_n . In order to state the result we introduce the nonnegative integer coefficients

$$d_n := n! \sum_{j=0}^n \frac{(-1)^j}{j!}, \quad n \in \mathbb{N}_0. \quad (12)$$

Note that d_n is the number of derangements (fixed point free permutations) of n elements. Clearly, $d_n = n d_{n-1} + (-1)^n$, $n \in \mathbb{N}$. For instance, $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, $d_3 = 2$, $d_4 = 9$, $d_5 = 44$, and $d_6 = 265$. A typical element of \mathcal{P}_i will be denoted by $\pi := \{B_1, \dots, B_l\}$, where $l := |\pi| \in \{1, \dots, i\}$ is the number of blocks of the partition and the blocks B_1, \dots, B_l of the partition are non-empty disjoint subsets of $\{1, \dots, i\}$ satisfying $B_1 \cup \dots \cup B_l = \{1, \dots, i\}$. Note that the order of the blocks is unimportant. For a set B and indices n_b , $b \in B$, we use in the following the notation $n_B := \sum_{b \in B} n_b$.

Theorem 3.1 (Alternative formula for the central moments) *A standard Gumbel distributed random variable G has central moments $m'_0 = 1$, $m'_1 = 0$ and*

$$m'_n = n! \sum_{i=1}^n \frac{1}{i!} \sum_{\substack{n_1, \dots, n_i \geq 2 \\ n_1 + \dots + n_i = n}} \frac{d_{n_1} \cdots d_{n_i}}{n_1! \cdots n_i!} s_i(n_1, \dots, n_i), \quad n \geq 2, \quad (13)$$

with coefficients d_n defined in (12) and $s_i(n_1, \dots, n_i)$ given via

$$s_i(n_1, \dots, n_i) := \sum_{\substack{k_1, \dots, k_i \in \mathbb{N} \\ \text{all distinct}}} \frac{1}{k_1^{n_1} \cdots k_i^{n_i}} \quad (14)$$

$$= \sum_{l=1}^i (-1)^{i-l} \sum_{\{B_1, \dots, B_l\} \in \mathcal{P}_i} (|B_1| - 1)! \cdots (|B_l| - 1)! \zeta(n_{B_1}) \cdots \zeta(n_{B_l}) \quad (15)$$

$$= \sum_{\pi \in \mathcal{P}_i} (-1)^{i-|\pi|} \prod_{B \in \pi} (|B| - 1)! \zeta(n_B), \quad (16)$$

where ζ denotes the zeta function and $|\pi|$ the number of blocks of the partition π .

Remarks. 1. The proof of Theorem 3.1 is based on the fact that the random variable S_n defined in (26) converges to $G - \gamma$ as $n \rightarrow \infty$. For more information on this convergence we refer the reader to the proof of Theorem 2.3.

2. Clearly, $s_i(n_1, \dots, n_i)$ is symmetric with respect to the entries n_1, \dots, n_i . For instance, $s_1(n_1) = \zeta(n_1)$, $s_2(n_1, n_2) = \zeta(n_1)\zeta(n_2) - \zeta(n_1 + n_2)$ and $s_3(n_1, n_2, n_3) = \zeta(n_1)\zeta(n_2)\zeta(n_3) - \zeta(n_1)\zeta(n_2 + n_3) - \zeta(n_2)\zeta(n_1 + n_3) - \zeta(n_3)\zeta(n_1 + n_2) + 2\zeta(n_1 + n_2 + n_3)$. A recursion for $s_i(n_1, \dots, n_i)$ is provided in (29). Eqs. (15) and (16) are reminiscent of sieve formulas, but we have not been able to rigorously relate these equations with some known sieve formula.

3. The values of the central moments m'_n for $0 \leq n \leq 10$ are listed in the appendix. The n th raw moment m_n of the Gumbel distribution is either obtained via (8) or (9), or from the central moments m'_j , $0 \leq j \leq n$, via (10). The book of Srivastava and Choi [12, pp. 370–371] contains the values of $(-1)^n m_n = \Gamma^{(n)}(1)$ for $1 \leq n \leq 10$.

4 Proofs

Proof. (of Proposition 2.1) For $n \in \mathbb{N}$ let T_n denote the absorption time of the Kingman coalescent restricted to a sample of size n . Clearly (see, for example, Kingman [8, Eq. (5.5)]), $T_n \stackrel{d}{=} \sum_{k=2}^n \tau_k$, where τ_2, τ_3, \dots are independent random variables and τ_k is exponentially distributed with parameter $\lambda_k = k(k-1)/2$, $k \geq 2$. Thus (see, for example, Ross [11, p. 309]), T_n has a hypoexponential distribution with density $g_n(t) := \sum_{k=2}^n a_{nk} \lambda_k e^{-\lambda_k t}$, $t > 0$, where

$$a_{nk} := \prod_{\substack{j=2 \\ j \neq k}}^n \frac{\lambda_j}{\lambda_j - \lambda_k} = (-1)^k (2k-1) \frac{n!(n-1)!}{(n-k)!(n+k-1)!}, \quad 2 \leq k \leq n.$$

Alternatively, the density g_n of T_n is obtained as follows. Let $D = (D_t)_{t \geq 0}$ denote the block counting process of the Kingman coalescent restricted to a sample of size n . From Lemma 5.1

(spectral decomposition) provided in the appendix it follows that T_n has distribution function

$$\begin{aligned}\mathbb{P}(T_n \leq t) &= \mathbb{P}(D_t = 1) = \sum_{k=1}^n e^{-\lambda_k t} r_{nk} l_{k1} = \sum_{k=1}^n e^{-\lambda_k t} \left(\prod_{j=k+1}^n \frac{\lambda_j}{\lambda_j - \lambda_k} \right) \left(\prod_{j=1}^{k-1} \frac{\lambda_{j+1}}{\lambda_j - \lambda_k} \right) \\ &= 1 - \sum_{k=2}^n e^{-\lambda_k t} \prod_{\substack{j=2 \\ j \neq k}}^n \frac{\lambda_j}{\lambda_j - \lambda_k} = 1 - \sum_{k=2}^n a_{nk} e^{-\lambda_k t}, \quad t \in [0, \infty),\end{aligned}$$

where the second last equality holds since $\lambda_1 = 0$. Taking the derivative with respect to t it follows that T_n has density g_n . In particular, T_n has moments

$$\mathbb{E}(T_n^j) = \int_0^\infty t^j g_n(t) dt = \sum_{k=2}^n a_{nk} \int_0^\infty t^j \lambda_k e^{-\lambda_k t} dt = j! \sum_{k=2}^n \frac{a_{nk}}{\lambda_k^j}, \quad j \in \mathbb{N}. \quad (17)$$

In the following it is shown, essentially by letting $n \rightarrow \infty$ in (17), that

$$\mathbb{E}(T^j) = j! \sum_{k=2}^\infty \frac{(-1)^k (2k-1)}{\lambda_k^j}, \quad j \in \mathbb{N}. \quad (18)$$

Eq. (18) holds for $j = 1$, since $\mathbb{E}(T) = 2$. Assume now that $j \geq 2$. From $T_n \nearrow T \stackrel{d}{=} \sum_{k=2}^\infty \tau_k$ almost surely as $n \rightarrow \infty$ it follows by monotone convergence that the left hand side in (17) converges to the left hand side in (18) as $n \rightarrow \infty$. In order to see that the right hand side in (17) converges to the right hand side in (18) as $n \rightarrow \infty$ fix $\varepsilon > 0$. Since $j \geq 2$ and $\lambda_k = k(k-1)/2$, the series $\sum_{k=2}^\infty (2k-1)/\lambda_k^j$ is absolutely convergent. Thus, there exists a constant $n_0 = n_0(\varepsilon)$ such that $\sum_{k=n_0+1}^\infty (2k-1)/\lambda_k^j < \varepsilon$. Noting that $a_{nk} = (-1)^k (2k-1) b_{nk}$ with

$$0 \leq b_{nk} := \frac{n!(n-1)!}{(n-k)!(n+k-1)!} = \frac{n(n-1) \cdots (n-k+1)}{(n+k-1)(n+k-2) \cdots n} \leq 1$$

it follows for all $n > n_0$ that

$$\begin{aligned}\left| \sum_{k=2}^n \frac{a_{nk}}{\lambda_k^j} - \sum_{k=2}^\infty \frac{(-1)^k (2k-1)}{\lambda_k^j} \right| &\leq \sum_{k=2}^n \frac{2k-1}{\lambda_k^j} \underbrace{|b_{nk} - 1|}_{\leq 1} + \sum_{k=n+1}^\infty \frac{2k-1}{\lambda_k^j} \\ &\leq \sum_{k=2}^{n_0} \frac{2k-1}{\lambda_k^j} |b_{nk} - 1| + \sum_{k=n_0+1}^\infty \frac{2k-1}{\lambda_k^j} \\ &\leq \sum_{k=2}^{n_0} \frac{2k-1}{\lambda_k^j} |b_{nk} - 1| + \varepsilon \rightarrow \varepsilon\end{aligned}$$

as $n \rightarrow \infty$, since $b_{nk} \rightarrow 1$ as $n \rightarrow \infty$ for each fixed $k \in \{2, 3, \dots\}$. Since $\varepsilon > 0$ can be chosen arbitrarily small, it follows that the right hand side in (17) converges to the right hand side in (18). Thus, (18) is established. Distinguishing in (18) even $k = 2m$ and odd $k = 2m+1$ and summing over all $m \in \mathbb{N}$ it follows that

$$\mathbb{E}(T^j) = j! \sum_{m=1}^\infty \frac{1}{m^j} \left(\frac{4m-1}{(2m-1)^j} - \frac{4m+1}{(2m+1)^j} \right), \quad j \in \mathbb{N},$$

and (2) is established. Let us now turn to the cumulants of T . Note that the exponential distribution with parameter $\lambda \in (0, \infty)$ has j th cumulant $(j-1)!/\lambda^j$. From $T_n \stackrel{d}{=} \sum_{k=2}^n \tau_k$ it follows that T_n has cumulants

$$\kappa_j(T_n) = \sum_{k=2}^n \kappa_j(\tau_k) = \sum_{k=2}^n \frac{(j-1)!}{\lambda_k^j} = (j-1)! 2^j \sum_{k=2}^n \frac{1}{k^j (k-1)^j}, \quad j \in \mathbb{N}. \quad (19)$$

We have verified above that the moments of T_n converge to those of T , which implies that the cumulants of T_n converge to those of T . Letting $n \rightarrow \infty$ in (19) yields (1).

Note that $\kappa_j(T)/(j-1)! = \sum_{k=2}^{\infty} 1/\lambda_k^j \rightarrow 1$ and that $\mathbb{E}(T^j)/j! = \sum_{k=2}^{\infty} (-1)^k (2k-1)/\lambda_k^j \rightarrow 3$ as $j \rightarrow \infty$, since $\lambda_2 = 1$ and $\lambda_k \geq \lambda_3 = 3$ for all $k \geq 3$.

It remains to verify the alternative formula (3) for the moments of T . For all $j \in \mathbb{N}$ we have

$$\begin{aligned} \mathbb{E}(T^j) &= \mathbb{E}\left(\left(\sum_{k=2}^{\infty} \tau_k\right)^j\right) = \sum_{k_1, \dots, k_j=2}^{\infty} \mathbb{E}(\tau_{k_1} \cdots \tau_{k_j}) \\ &= \sum_{k_1, \dots, k_j=2}^{\infty} \prod_{m=2}^{\infty} \mathbb{E}(\tau_m^{a_m}) = \sum_{2 \leq k_1 \leq \dots \leq k_j} \frac{j!}{\prod_{m=2}^{\infty} a_m!} \prod_{m=2}^{\infty} \mathbb{E}(\tau_m^{a_m}), \end{aligned}$$

where, for $m \geq 2$, a_m denotes the number of indices k_1, \dots, k_j being equal to m . Note that $\sum_{m=2}^{\infty} a_m = j$. Since $\mathbb{E}(\tau_m^{a_m}) = a_m!/\lambda_m^{a_m}$, the above expression simplifies to

$$\mathbb{E}(T^j) = j! \sum_{2 \leq k_1 \leq \dots \leq k_j} \frac{1}{\lambda_{k_1} \cdots \lambda_{k_j}} = j! 2^j \sum_{2 \leq k_1 \leq \dots \leq k_j} \prod_{i=1}^j \frac{1}{k_i (k_i - 1)},$$

which is (3). \square

The proof of Theorem 2.2 is based on the following basic but fundamental lemma, which provides a solution for a certain two dimensional recursion with a countable number of initial values.

Lemma 4.1 *Let $a_1, b_1, a_2, b_2, \dots \in \mathbb{R}$. For $i, j \in \mathbb{N}$ define s_{ij} recursively via $s_{ij} := s_{i-1,j} - s_{i,j-1}$, with initial values $s_{0k} := a_k$ and $s_{k0} := b_k$, $k \in \mathbb{N}$. Then*

$$s_{ij} = \sum_{k=1}^j (-1)^{j-k} \binom{i+j-k-1}{i-1} a_k + (-1)^j \sum_{k=1}^i \binom{i+j-k-1}{j-1} b_k, \quad i, j \in \mathbb{N}. \quad (20)$$

Proof. (of Lemma 4.1) Induction on $i+j$. Clearly, (20) holds for $i+j = 2$, since $s_{11} = s_{01} - s_{10} = a_1 - b_1$. For the induction step from $i+j-1$ to $i+j$ (> 2) three cases are distinguished.

Case 1: If $j = 1$, then $i > 1$ and, hence, by the recursion and by induction, $s_{i1} = s_{i-1,1} - s_{i0} = (a_1 - \sum_{k=1}^{i-1} b_k) - b_i = a_1 - \sum_{k=1}^i b_k$, which is (20) for $j = 1$.

Case 2: If $i = 1$, then $j > 1$ and, hence, by the recursion and by induction, $s_{1j} = s_{0j} - s_{1,j-1} = a_j - (\sum_{k=1}^{j-1} (-1)^{j-1-k} a_k + (-1)^{j-1} b_1) = \sum_{k=1}^j (-1)^{j-k} a_k + (-1)^j b_1$, which is (20) for $i = 1$.

Case 3: If $i, j > 1$ then

$$\begin{aligned}
s_{ij} &= s_{i-1,j} - s_{i,j-1} \\
&= \sum_{k=1}^j (-1)^{j-k} \binom{i+j-k-2}{i-2} a_k + (-1)^j \sum_{k=1}^{i-1} \binom{i+j-k-2}{j-1} b_k \\
&\quad - \sum_{k=1}^{j-1} (-1)^{j-1-k} \binom{i+j-k-2}{i-1} a_k - (-1)^{j-1} \sum_{k=1}^i \binom{i+j-k-2}{j-2} b_k \\
&= a_j + \sum_{k=1}^{j-1} (-1)^{j-k} \left(\binom{i+j-k-2}{i-2} + \binom{i+j-k-2}{i-1} \right) a_k \\
&\quad + (-1)^j b_i + (-1)^j \sum_{k=1}^{i-1} \left(\binom{i+j-k-2}{j-2} + \binom{i+j-k-2}{j-1} \right) b_k \\
&= \sum_{k=1}^j (-1)^{j-k} \binom{i+j-k-1}{i-1} a_k + (-1)^j \sum_{k=1}^i \binom{i+j-k-1}{j-1} b_k,
\end{aligned}$$

which completes the induction. \square

Before we come to the proof of Theorem 2.2 we provide a typical application of Lemma 4.1 showing that this lemma can be used to determine the value of certain series.

Example. For $j \in \mathbb{N}$ we would like to determine the series $\sum_{k=2}^{\infty} 1/(k^j(k-1)^j)$. We proceed as follows. For $i, j \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ with $i+j \geq 2$ define $s_{ij} := \sum_{k=2}^{\infty} 1/(k^i(k-1)^j)$. Note that $s_{11} = 1$, $s_{i0} = \zeta(i) - 1$, $i \geq 2$, and that $s_{0j} = \zeta(j)$, $j \geq 2$. For all $i, j \in \mathbb{N}$ with $i+j \geq 3$,

$$s_{ij} = \sum_{k=2}^{\infty} \frac{1}{k^{i-1}(k-1)^{j-1}} \frac{1}{k(k-1)} = \sum_{k=2}^{\infty} \frac{1}{k^{i-1}(k-1)^{j-1}} \left(\frac{1}{k-1} - \frac{1}{k} \right) = s_{i-1,j} - s_{i,j-1}.$$

If we additionally define $s_{01} := 1$ and $s_{10} := 0$, then this recursion holds also for $i = j = 1$, so for all $i, j \in \mathbb{N}$. By Lemma 4.1, for all $j \in \mathbb{N}$,

$$\begin{aligned}
s_{jj} &= \sum_{k=1}^j (-1)^{j-k} \binom{2j-k-1}{j-1} s_{0k} + (-1)^j \sum_{k=1}^j \binom{2j-k-1}{j-1} s_{k0} \\
&= (-1)^j \sum_{\substack{k=2 \\ k \text{ even}}}^j \binom{2j-k-1}{j-1} (s_{0k} + s_{k0}) + (-1)^j \sum_{\substack{k=1 \\ k \text{ odd}}}^j \binom{2j-k-1}{j-1} (s_{k0} - s_{0k}).
\end{aligned}$$

Plugging in $s_{0k} + s_{k0} = 2\zeta(k) - 1$ for even k and $s_{k0} - s_{0k} = -1$ for odd k we obtain the solution

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{1}{k^j(k-1)^j} &= s_{jj} = (-1)^{j+1} \sum_{k=1}^j \binom{2j-k-1}{j-1} + 2(-1)^j \sum_{\substack{k=2 \\ k \text{ even}}}^j \binom{2j-k-1}{j-1} \zeta(k) \\
&= (-1)^{j+1} \binom{2j-1}{j} + 2(-1)^j \sum_{m=1}^{\lfloor j/2 \rfloor} \binom{2j-2m-1}{j-1} \zeta(2m)
\end{aligned}$$

$$= 2(-1)^j \sum_{m=0}^{\lfloor j/2 \rfloor} \binom{2j-2m-1}{j-1} \zeta(2m), \quad j \in \mathbb{N}, \quad (21)$$

since $\zeta(0) = -1/2$. For instance $s_{22} = 2\zeta(2) - 3 = \pi^2/3 - 3 \approx 0.28987$ and $s_{33} = 10 - 6\zeta(2) = 10 - \pi^2 \approx 0.13040$.

The following proof of Theorem 2.2 has much in common with the previous example, but a modified double sequence $(s_{ij})_{i,j \in \mathbb{N}}$ is used having in particular more involved initial values.

Proof. (of Theorem 2.2) The formula (4) for the cumulants of T follows directly from (1) by multiplying (21) with $(j-1)!2^j$. In order to verify the formula (5) for the moments of T define for $i, j \in \mathbb{N}_0$ with $i+j \geq 2$

$$s_{ij} := \sum_{k=2}^{\infty} \frac{(-1)^k (2k-1)}{k^i (k-1)^j}.$$

By Proposition 2.1, $\mathbb{E}(T^j) = j!2^j s_{jj}$. Thus it remains to verify that

$$s_{jj} = 2(-1)^j \sum_{m=0}^{\lfloor j/2 \rfloor} (2m-1) \left(1 - \frac{1}{2^{2m-1}}\right) \frac{(2j-2m-2)!}{(j-1)!(j-2m)!} \zeta(2m), \quad j \in \mathbb{N}. \quad (22)$$

It is straightforward to check that $s_{11} = 1$, $s_{02} = 2 \log 2 + \zeta(2)/2$, $s_{20} = 1 - 2 \log 2 + \zeta(2)/2$,

$$s_{i0} = 1 - 2 \left(1 - \frac{1}{2^{i-2}}\right) \zeta(i-1) + \left(1 - \frac{1}{2^{i-1}}\right) \zeta(i), \quad i \geq 3,$$

and

$$s_{0j} = 2 \left(1 - \frac{1}{2^{j-2}}\right) \zeta(j-1) + \left(1 - \frac{1}{2^{j-1}}\right) \zeta(j), \quad j \geq 3.$$

For all $i, j \in \mathbb{N}$ with $i+j \geq 3$ we have

$$s_{ij} = \sum_{k=2}^{\infty} \frac{(-1)^k (2k-1)}{k^{i-1} (k-1)^{j-1}} \frac{1}{k(k-1)} = \sum_{k=2}^{\infty} \frac{(-1)^k (2k-1)}{k^{i-1} (k-1)^{j-1}} \left(\frac{1}{k-1} - \frac{1}{k} \right) = s_{i-1,j} - s_{i,j-1}. \quad (23)$$

If we additionally define $s_{01} := 1$ and $s_{10} := 0$, then the recursion (23) holds also for $i = j = 1$, so for all $i, j \in \mathbb{N}$. By Lemma 4.1,

$$s_{jj} = \sum_{k=1}^j (-1)^{j-k} \binom{2j-k-1}{j-1} s_{0k} + (-1)^j \sum_{k=1}^j \binom{2j-k-1}{j-1} s_{k0}, \quad j \in \mathbb{N}.$$

Ordering with respect to even and odd k yields

$$(-1)^j s_{jj} = \sum_{\substack{k=2 \\ k \text{ even}}}^j \binom{2j-k-1}{j-1} (s_{0k} + s_{k0}) + \sum_{\substack{k=1 \\ k \text{ odd}}}^j \binom{2j-k-1}{j-1} (s_{k0} - s_{0k}).$$

Plugging in $s_{0k} + s_{k0} = 1 + 2(1 - 1/2^{k-1})\zeta(k)$ for even k , $s_{10} - s_{01} = -1$, and $s_{k0} - s_{0k} = 1 - 4(1 - 1/2^{k-2})\zeta(k-1)$ for odd $k \geq 3$, it follows that

$$\begin{aligned}
(-1)^j s_{jj} &= \sum_{\substack{k=2 \\ k \text{ even}}}^j \binom{2j-k-1}{j-1} \left(1 + 2\left(1 - \frac{1}{2^{k-1}}\right)\zeta(k)\right) \\
&\quad + \binom{2j-2}{j-1}(-1) + \sum_{\substack{k=3 \\ k \text{ odd}}}^j \binom{2j-k-1}{j-1} \left(1 - 4\left(1 - \frac{1}{2^{k-2}}\right)\zeta(k-1)\right) \\
&= \sum_{k=2}^j \binom{2j-k-1}{j-1} - \binom{2j-2}{j-1} + 2 \sum_{\substack{k=2 \\ k \text{ even}}}^j \binom{2j-k-1}{j-1} \left(1 - \frac{1}{2^{k-1}}\right)\zeta(k) \\
&\quad - 4 \sum_{\substack{k=3 \\ k \text{ odd}}}^j \binom{2j-k-1}{j-1} \left(1 - \frac{1}{2^{k-2}}\right)\zeta(k-1) \\
&= \binom{2j-2}{j} - \binom{2j-2}{j-1} + 2 \sum_{\substack{k=2 \\ k \text{ even}}}^j \binom{2j-k-1}{j-1} \left(1 - \frac{1}{2^{k-1}}\right)\zeta(k) \\
&\quad - 4 \sum_{\substack{k=2 \\ k \text{ even}}}^{j-1} \binom{2j-k-2}{j-1} \left(1 - \frac{1}{2^{k-1}}\right)\zeta(k).
\end{aligned}$$

Thus,

$$(-1)^j s_{jj} = c_j + \sum_{\substack{k=2 \\ k \text{ even}}}^j c_{jk}\zeta(k), \quad (24)$$

where $c_j := \binom{2j-2}{j} - \binom{2j-2}{j-1} = -(2j-2)!/(j!(j-1)!)$ and

$$\begin{aligned}
c_{jk} &:= 2\left(1 - \frac{1}{2^{k-1}}\right) \left(\binom{2j-k-1}{j-1} - 2\binom{2j-k-2}{j-1} \right) \\
&= 2\left(1 - \frac{1}{2^{k-1}}\right) \left(\frac{(2j-k-1)!}{(j-1)!(j-k)!} - 2\frac{(2j-k-2)!}{(j-1)!(j-k-1)!} \right) \\
&= 2\left(1 - \frac{1}{2^{k-1}}\right) \frac{(2j-k-2)!}{(j-1)!(j-k)!} ((2j-k-1) - 2(j-k)) \\
&= 2(k-1) \left(1 - \frac{1}{2^{k-1}}\right) \frac{(2j-k-2)!}{(j-1)!(j-k)!}.
\end{aligned}$$

Substituting $k = 2m$ in (24) and noting that $\zeta(0) = -1/2$ yields (22). The proof is complete. \square

In order to prepare the proof of Theorem 2.3 recall that a standard Gumbel distributed random variable G has (see, for example, [5, p. 12, Eqs. (22.29) and (22.30)]) cumulants $\kappa_1(G) = \gamma$ (Euler's constant) and $\kappa_j(G) = (-1)^j \Psi^{(j-1)}(1) = (j-1)! \zeta(j)$, $j \geq 2$.

Proof. (of Theorem 2.3) We start similar as in the proof of Proposition 2.1. Clearly, $L_n \stackrel{d}{=} \sum_{k=2}^n k\tau_k = \sum_{k=2}^n X_k$, where the random variables $X_k := k\tau_k$, $k \geq 2$, are independent and X_k is

exponentially distributed with parameter $\mu_k := \lambda_k/k = (k-1)/2$, $k \geq 2$. Thus, L_n has cumulants

$$\kappa_j(L_n) = \sum_{k=2}^n \kappa_j(X_k) = \sum_{k=2}^n \frac{(j-1)!}{\mu_k^j} = (j-1)! 2^j \sum_{k=2}^n \frac{1}{(k-1)^j}, \quad j \in \mathbb{N}. \quad (25)$$

For $j = 1$ we have $\kappa_1(G_n) = \kappa_1(L_n/2 - \log n) = \kappa_1(L_n)/2 - \log n = \sum_{k=2}^n 1/(k-1) - \log n \rightarrow \gamma = \kappa_1(G)$ as $n \rightarrow \infty$. For $j \geq 2$ we have $\kappa_j(G_n) = \kappa_j(L_n)/2^j = (j-1)! \sum_{k=2}^n 1/(k-1)^j \rightarrow (j-1)! \zeta(j) = \kappa_j(G)$ as $n \rightarrow \infty$. Thus, we have convergence $\kappa_j(G_n) \rightarrow \kappa_j(G)$ as $n \rightarrow \infty$ of all cumulants, which implies the convergence $\mathbb{E}(G_n^j) \rightarrow \mathbb{E}(G^j)$ as $n \rightarrow \infty$ of all moments.

For the convergence $G_n := L_n/2 - \log n \rightarrow G$ in distribution as $n \rightarrow \infty$ we refer the reader to [3, Lemma 7.1] and the references in the remark thereafter. In order to verify that the convergence $G_n \rightarrow G$ holds even almost surely define

$$S_n := \frac{L_n - \mathbb{E}(L_n)}{2} = \sum_{k=2}^n \frac{X_k - \mathbb{E}(X_k)}{2} = \sum_{k=2}^n Y_k, \quad n \in \mathbb{N}, \quad (26)$$

where $Y_k := (X_k - \mathbb{E}(X_k))/2$, $k \geq 2$. Note that Y_2, Y_3, \dots are independent. From $\mathbb{E}(L_n) = 2 \sum_{k=2}^n 1/(k-1) = 2 \log n + 2\gamma + O(1/n)$ we conclude that $S_n \rightarrow G - \gamma$ in distribution as $n \rightarrow \infty$. It is well known (see, for example [1, Theorem 22.7]) that a sum $S_n = \sum_{k=2}^n Y_k$ of independent random variables converges in distribution if and only if it converges almost surely. Thus we even have $S_n \rightarrow G - \gamma$ almost surely. Thus, $G_n \rightarrow G$ almost surely as $n \rightarrow \infty$. By [6, Proposition 3.12] it follows that $G_n \rightarrow G$ in L^p for any $p \in (0, \infty)$, and the sequence $(G_n^p)_{n \in \mathbb{N}}$ is uniformly integrable for any $p \in (0, \infty)$. Note that $\|S_n - (G - \gamma)\|_p \leq \|\gamma + S_n - G_n\|_p + \|G_n - G\|_p = \|\gamma + \log n - \sum_{k=2}^n 1/(k-1)\|_p + \|G_n - G\|_p \rightarrow 0$ as $n \rightarrow \infty$, so we also have $S_n \rightarrow G - \gamma$ in L^p for any $p \in (0, \infty)$ and, hence, convergence $\mathbb{E}(S_n^j) \rightarrow \mathbb{E}((G - \gamma)^j)$ as $n \rightarrow \infty$ of all moments. Furthermore, $(S_n)_{n \in \mathbb{N}}$ is a martingale, but we did not use this property in the proof. \square

Remark. In this remark formulas for the moments of the total tree length L_n are provided. It is known (see, for example, [3, Lemma 7.1] and the remark thereafter) that L_n has the same distribution as the maximum of $n-1$ independent and exponentially distributed random variables with parameter $1/2$. In particular, L_n has distribution function $\mathbb{P}(L_n \leq t) = (1 - e^{-t/2})^{n-1}$, $t > 0$, and, hence, moments

$$\begin{aligned} \mathbb{E}(L_n^j) &= \int_0^\infty j t^{j-1} \mathbb{P}(L_n > t) dt = \int_0^\infty j t^{j-1} (1 - (1 - e^{-t/2})^{n-1}) dt \\ &= \int_0^\infty -j t^{j-1} \sum_{k=1}^{n-1} \binom{n-1}{k} (-e^{-t/2})^k dt = \sum_{k=1}^{n-1} (-1)^{k+1} \binom{n-1}{k} \int_0^\infty j t^{j-1} e^{-kt/2} dt \\ &= \sum_{k=1}^{n-1} (-1)^{k+1} \binom{n-1}{k} \frac{j!}{(k/2)^j} = j! 2^j \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k^j} \binom{n-1}{k}, \quad j \in \mathbb{N}. \end{aligned} \quad (27)$$

Alternatively,

$$\begin{aligned} \mathbb{E}(L_n^j) &= \mathbb{E}\left(\left(\sum_{k=2}^n X_k\right)^j\right) = \sum_{k_1, \dots, k_j=2}^n \mathbb{E}(X_{k_1} \cdots X_{k_j}) \\ &= \sum_{k_1, \dots, k_j=2}^n \mathbb{E}(X_2^{a_2}) \cdots \mathbb{E}(X_n^{a_n}) = \sum_{2 \leq k_1 \leq \dots \leq k_j \leq n} \frac{j!}{a_2! \cdots a_n!} \mathbb{E}(X_2^{a_2}) \cdots \mathbb{E}(X_n^{a_n}), \end{aligned}$$

where, for $m \in \{2, \dots, n\}$, a_m denotes the number of indices k_1, \dots, k_j being equal to m . Since $\mathbb{E}(X_m^{a_m}) = a_m!/\mu_k^{a_m}$, the above expression simplifies to

$$\mathbb{E}(L_n^j) = j! \sum_{2 \leq k_1 \leq \dots \leq k_j \leq n} \frac{1}{\mu_{k_1} \dots \mu_{k_j}} = j! 2^j \sum_{1 \leq k_1 \leq \dots \leq k_j \leq n-1} \frac{1}{k_1 \dots k_j}, \quad j \in \mathbb{N}. \quad (28)$$

Comparing (27) with (28) leads to the combinatorial identity

$$\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k^j} \binom{n-1}{k} = \sum_{1 \leq k_1 \leq \dots \leq k_j \leq n-1} \frac{1}{k_1 \dots k_j}, \quad j \in \mathbb{N}, n \geq 2.$$

Proof. (of Theorem 3.1) Note first that an exponentially distributed random variable X with parameter $\alpha \in (0, \infty)$ has moments $\mathbb{E}(X^n) = n!/\alpha^n$ and central moments $\mathbb{E}((X - \mathbb{E}(X))^n) = d_n/\alpha^n$ with d_n defined in (12), $n \in \mathbb{N}_0$. Consider the random variable S_N defined via (26). From the proof of Theorem 2.3 it is already known that $S_N \rightarrow G - \gamma$ almost surely as $N \rightarrow \infty$ with convergence of all moments. Moreover, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} \mathbb{E}(S_N^n) &= \mathbb{E}((Y_2 + \dots + Y_N)^n) \\ &= \sum_{i=1}^n \sum_{2 \leq k_1 < \dots < k_i \leq N} \sum_{\substack{n_1, \dots, n_i \geq 1 \\ n_1 + \dots + n_i = n}} \frac{n!}{n_1! \dots n_i!} \mathbb{E}(Y_{k_1}^{n_1}) \dots \mathbb{E}(Y_{k_i}^{n_i}) \\ &= \sum_{i=1}^n \frac{1}{i!} \sum_{\substack{k_1, \dots, k_i=2 \\ \text{all distinct}}}^N \sum_{\substack{n_1, \dots, n_i \geq 1 \\ n_1 + \dots + n_i = n}} \frac{n!}{n_1! \dots n_i!} \mathbb{E}(Y_{k_1}^{n_1}) \dots \mathbb{E}(Y_{k_i}^{n_i}) \\ &= n! \sum_{i=1}^n \frac{1}{i!} \sum_{\substack{n_1, \dots, n_i \geq 1 \\ n_1 + \dots + n_i = n}} \frac{1}{n_1! \dots n_i!} \sum_{\substack{k_1, \dots, k_i=2 \\ \text{all distinct}}}^N \frac{d_{n_1}}{(k_1 - 1)^{n_1}} \dots \frac{d_{n_i}}{(k_i - 1)^{n_i}} \\ &= n! \sum_{i=1}^n \frac{1}{i!} \sum_{\substack{n_1, \dots, n_i \geq 2 \\ n_1 + \dots + n_i = n}} \frac{d_{n_1} \dots d_{n_i}}{n_1! \dots n_i!} \sum_{\substack{k_1, \dots, k_i=1 \\ \text{all distinct}}}^{N-1} \frac{1}{k_1^{n_1} \dots k_i^{n_i}}, \end{aligned}$$

since $d_1 = 0$. Letting $N \rightarrow \infty$ it follows that G has central moments (13) with $s_i(n_1, \dots, n_i)$ defined via (14). Obviously, the expressions in (15) and (16) coincide. Thus, it remains to verify that $s_i(n_1, \dots, n_i)$ can be expressed in terms of the zeta function via (15). We show this by induction on $i \in \mathbb{N}$. Clearly, (15) holds for $i = 1$, since $s_1(n_1) = \zeta(n_1)$ for all $n_1 \geq 2$. Concerning the induction step from $1, \dots, i$ to $i + 1$ (≥ 2) note first that, for all $i \in \mathbb{N}$ and all $n_1, \dots, n_i \geq 2$,

$$\begin{aligned} s_{i+1}(n_1, \dots, n_{i+1}) &= \sum_{\substack{k_1, \dots, k_i \in \mathbb{N} \\ \text{all distinct}}} \frac{1}{k_1^{n_1} \dots k_i^{n_i}} \sum_{k_{i+1} \in \mathbb{N} \setminus \{k_1, \dots, k_i\}} \frac{1}{k_{i+1}^{n_{i+1}}} \\ &= \sum_{\substack{k_1, \dots, k_i \in \mathbb{N} \\ \text{all distinct}}} \frac{1}{k_1^{n_1} \dots k_i^{n_i}} \left(\sum_{k_{i+1} \in \mathbb{N}} \frac{1}{k_{i+1}^{n_{i+1}}} - \sum_{r=1}^i \frac{1}{k_r^{n_{i+1}}} \right) \\ &= s_i(n_1, \dots, n_i) s_1(n_{i+1}) - \sum_{r=1}^i s_i(n_1, \dots, n_{r-1}, n_r + n_{i+1}, n_{r+1}, \dots, n_i). \end{aligned} \quad (29)$$

By induction we conclude that

$$\begin{aligned}
& s_i(n_1, \dots, n_i) s_1(n_{i+1}) \\
&= \sum_{l=1}^i (-1)^{i-l} \sum_{\{B_1, \dots, B_l\} \in \mathcal{P}_i} (|B_1| - 1)! \cdots (|B_l| - 1)! \zeta(n_{B_1}) \cdots \zeta(n_{B_l}) \zeta(n_{i+1}) \\
&= \sum_{l=1}^i (-1)^{i-l} \sum_{\substack{\{B_1, \dots, B_{l+1}\} \in \mathcal{P}_{i+1} \\ B_{l+1} = \{i+1\}}} (|B_1| - 1)! \cdots (|B_{l+1}| - 1)! \zeta(n_{B_1}) \cdots \zeta(n_{B_{l+1}}) \\
&= \sum_{l=2}^{i+1} (-1)^{i+1-l} \sum_{\substack{\{B_1, \dots, B_l\} \in \mathcal{P}_{i+1} \\ B_l = \{i+1\}}} (|B_1| - 1)! \cdots (|B_l| - 1)! \zeta(n_{B_1}) \cdots \zeta(n_{B_l}). \tag{30}
\end{aligned}$$

Also by induction it is seen that

$$\begin{aligned}
& \sum_{r=1}^i s_i(n_1, \dots, n_{r-1}, n_r + n_{i+1}, n_{r+1}, \dots, n_i) \\
&= \sum_{r=1}^i \sum_{l=1}^i (-1)^{i-l} \sum_{\substack{\{A_1, \dots, A_l\} \in \mathcal{P}_i \\ r \in A_l}} (|A_1| - 1)! \cdots (|A_l| - 1)! \zeta(n_{A_1}) \cdots \zeta(n_{A_{l-1}}) \zeta(n_{A_l \cup \{i+1\}}),
\end{aligned}$$

where, in the last sum, we ordered without loss of generality the blocks A_1, \dots, A_l of the partition such that the element r belongs to the last block A_l . Reordering the sums on the right hand side yields

$$\begin{aligned}
& \sum_{r=1}^i s_i(n_1, \dots, n_{r-1}, n_r + n_{i+1}, n_{r+1}, \dots, n_i) \\
&= \sum_{l=1}^i (-1)^{i-l} \sum_{\{A_1, \dots, A_l\} \in \mathcal{P}_i} \sum_{\substack{r=1 \\ r \in A_l}}^i (|A_1| - 1)! \cdots (|A_l| - 1)! \zeta(n_{A_1}) \cdots \zeta(n_{A_{l-1}}) \zeta(n_{A_l \cup \{i+1\}}).
\end{aligned}$$

Rewriting this expression in terms of the blocks $B_1 := A_1, \dots, B_{l-1} := A_{l-1}$ and $B_l := A_l \cup \{i+1\}$ it follows that

$$\begin{aligned}
& \sum_{r=1}^i s_i(n_1, \dots, n_{r-1}, n_r + n_{i+1}, n_{r+1}, \dots, n_i) \\
&= \sum_{l=1}^i (-1)^{i-l} \sum_{\{B_1, \dots, B_l\} \in \mathcal{P}_{i+1}} \sum_{\substack{r=1 \\ r, i+1 \in B_l}}^i (|B_1| - 1)! \cdots (|B_{l-1}| - 1)! (|B_l| - 2)! \zeta(n_{B_1}) \cdots \zeta(n_{B_l}).
\end{aligned}$$

The last sum (over r) consists of $|B_l| - 1$ summands and these summands do not depend on r , which gives rise to a factor $|B_l| - 1$ leading to

$$\sum_{r=1}^i s_i(n_1, \dots, n_{r-1}, n_r + n_{i+1}, n_{r+1}, \dots, n_i)$$

$$= \sum_{l=1}^i (-1)^{i-l} \sum_{\substack{\{B_1, \dots, B_l\} \in \mathcal{P}_{i+1} \\ B_k \neq \{i+1\} \text{ for all } k \in \{1, \dots, l\}}} (|B_1| - 1)! \cdots (|B_l| - 1)! \zeta(n_{B_1}) \cdots \zeta(n_{B_l}). \quad (31)$$

Subtracting (31) from (30) and recalling (29) it follows that

$$s_{i+1}(n_1, \dots, n_{i+1}) = \sum_{l=1}^{i+1} (-1)^{i+1-l} \sum_{\{B_1, \dots, B_l\} \in \mathcal{P}_{i+1}} (|B_1| - 1)! \cdots (|B_l| - 1)! \zeta(n_{B_1}) \cdots \zeta(n_{B_l}),$$

which completes the induction. Thus, (15) is established. \square

5 Appendix

Central moments of the Gumbel distribution

For completeness we record the central moment m'_n of the Gumbel distribution for $0 \leq n \leq 10$. Based on (13) we provide them in terms of the coefficients $s_i(n_1, \dots, n_i)$, in terms of the zeta function, and numerically. The first central moments of the Gumbel distribution are $m'_0 = 1$, $m'_1 = 0$, $m'_2 = s_1(2) = \zeta(2) = \pi^2/6 \approx 1.64493$, $m'_3 = 2s_1(3) = 2\zeta(3) \approx 2.40411$, $m'_4 = 9s_1(4) + 3s_2(2, 2) = 6\zeta(4) + 3\zeta^2(2) = 3\pi^4/20 \approx 14.61136$, $m'_5 = 44s_1(5) + 20s_2(2, 3) = 24\zeta(5) + 20\zeta(2)\zeta(3) = 24\zeta(5) + (10/3)\pi^2\zeta(3) \approx 64.43235$,

$$\begin{aligned} m'_6 &= 265s_1(6) + 135s_2(2, 4) + 40s_2(3, 3) + 15s_3(2, 2, 2) \\ &= 120\zeta(6) + 90\zeta(2)\zeta(4) + 40\zeta^2(3) + 15\zeta^3(2) \\ &= \frac{61}{168}\pi^6 + 40\zeta^2(3) \approx 406.87347, \end{aligned}$$

$$\begin{aligned} m'_7 &= 1854s_1(7) + 924s_2(2, 5) + 630s_2(3, 4) + 210s_3(2, 2, 3) \\ &= 720\zeta(7) + 504\zeta(2)\zeta(5) + 420\zeta(3)\zeta(4) + 210\zeta^2(2)\zeta(3) \\ &= 720\zeta(7) + 84\pi^2\zeta(5) + \frac{21}{2}\pi^4\zeta(3) \approx 2815.13142, \end{aligned}$$

$$\begin{aligned} m'_8 &= 14833s_1(8) + 7420s_2(2, 6) + 4928s_2(3, 5) + 2835s_2(4, 4) \\ &\quad + 1890s_3(2, 2, 4) + 1120s_3(2, 3, 3) + 105s_4(2, 2, 2, 2) \\ &= 5040\zeta(8) + 3360\zeta(2)\zeta(6) + 2688\zeta(3)\zeta(5) + 1260\zeta^2(4) \\ &\quad + 1260\zeta^2(2)\zeta(4) + 1120\zeta(2)\zeta^2(3) + 105\zeta^4(2) \\ &= \frac{1261}{720}\pi^8 + 2688\zeta(3)\zeta(5) + \frac{560}{3}\pi^2\zeta^2(3) \approx 22630.60731, \end{aligned}$$

$$\begin{aligned}
m'_9 &= 133496s_1(9) + 66744s_2(2, 7) + 44520s_2(3, 6) + 49896s_2(4, 5) \\
&\quad + 16632s_3(2, 2, 5) + 22680s_3(2, 3, 4) + 2240s_3(3, 3, 3) + 2520s_4(2, 2, 2, 3) \\
&= 40320\zeta(9) + 25920\zeta(2)\zeta(7) + 20160\zeta(3)\zeta(6) + 18144\zeta(4)\zeta(5) \\
&\quad + 9072\zeta^2(2)\zeta(5) + 15120\zeta(2)\zeta(3)\zeta(4) + 2240\zeta^3(3) + 2520\zeta^3(2)\zeta(3) \\
&= 40320\zeta(9) + 4320\pi^2\zeta(7) + \frac{2268}{5}\pi^4\zeta(5) + 2240\zeta^3(3) + 61\pi^6\zeta(3) \\
&\approx 203595.03670, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
m'_{10} &= 1334961s_1(10) + 667485s_2(2, 8) + 444960s_2(3, 7) + 500850s_2(4, 6) \\
&\quad + 243936s_2(5, 5) + 166950s_3(2, 2, 6) + 221760s_3(2, 3, 5) + 127575s_3(2, 4, 4) \\
&\quad + 75600s_3(3, 3, 4) + 28350s_4(2, 2, 2, 4) + 25200s_4(2, 2, 3, 3) + 945s_5(2, 2, 2, 2, 2) \\
&= 362880\zeta(10) + 226800\zeta(2)\zeta(8) + 172800\zeta(3)\zeta(7) + 151200\zeta(4)\zeta(6) \\
&\quad + 72576\zeta^2(5) + 75600\zeta^2(2)\zeta(6) + 120960\zeta(2)\zeta(3)\zeta(5) + 56700\zeta(2)\zeta^2(4) \\
&\quad + 50400\zeta^2(3)\zeta(4) + 18900\zeta^3(2)\zeta(4) + 25200\zeta^2(2)\zeta^2(3) + 945\zeta^5(2) \\
&= \frac{4977}{352}\pi^{10} + 172800\zeta(3)\zeta(7) + 72576\zeta^2(5) + 20160\pi^2\zeta(3)\zeta(5) + 1260\pi^4\zeta^2(3) \\
&\approx 2036946.09776.
\end{aligned}$$

A spectral decomposition

Lemma 5.1 (Spectral decomposition of a pure death process) *Let $n \in \mathbb{N}$ and let $X := (X_t)_{t \geq 0}$ be a pure death process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $\{1, \dots, n\}$ and pairwise distinct death rates d_1, \dots, d_n . Then, the transition probabilities $p_{ij}(t) := \mathbb{P}(X_t = j \mid X_0 = i)$ are given by*

$$p_{ij}(t) = \sum_{k=j}^i e^{-d_k t} r_{ik} l_{kj}, \quad i, j \in \{1, \dots, n\}, \quad (32)$$

where the $n \times n$ matrices $R = (r_{ij})$ and $L = (l_{ij})$ are defined via $r_{ij} := l_{ij} := 0$ for $i < j$ and

$$r_{ij} := \prod_{l=j+1}^i \frac{d_l}{d_l - d_j} \quad \text{and} \quad l_{ij} := \prod_{l=j}^{i-1} \frac{d_{l+1}}{d_l - d_i} \quad \text{for } i \geq j.$$

Proof. With some effort it can be checked that $RL = I$, where $I = (\delta_{ij})_{1 \leq i, j \leq n}$ denotes the $n \times n$ unit matrix. Let $Q = (q_{ij})_{1 \leq i, j \leq n}$ denote the generator matrix of X , i.e. $q_{ii} := -d_i$, $q_{i, i-1} := d_i$ and $q_{ij} = 0$ otherwise. Furthermore, let D denote the diagonal matrix with entries $d_{ij} := -d_i$ for $i = j$ and $d_{ij} := 0$ otherwise. It is readily checked that $RD = QR$, and, hence, $RDL = Q$. The transition probabilities $p_{ij}(t)$ of the process X are now obtained from the spectral decomposition $P(t) := e^{tQ} = e^{tRDL} = R(e^{tD})L$ of the transition matrix $P(t)$ as $p_{ij}(t) = \sum_{k=1}^n e^{-d_k t} r_{ik} l_{kj} = \sum_{k=j}^i e^{-d_k t} r_{ik} l_{kj}$. \square

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